Implications of a law of a decelerating logarithmic growth in mathematics and reading achievement

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Abstract

The study of growth in educational achievement has relied largely on correlational and descriptive modelling, methods typical in the social sciences. Using these methods, researchers have concluded that the rate of growth in achievement in mathematics and reading is greater in earlier than later years of schooling, and that achievements in the two periods are related. However, there is no single mathematical equation that distils the essence of the relationship between the variables, the kind of relationship found in the natural sciences. Instead, there are multiple descriptive models used to summarise the different rates of growth at different times of schooling. Using educational achievements placed on a single quantitative scale of modern test theory, this paper first demonstrates that growth in means in both mathematics and reading achievement of different large cohorts of students from different countries and different jurisdictions within countries, decelerates virtually perfectly logarithmically. It then establishes that as a result, the rate of growth on a quantitative scale is inversely proportional to the time spent in formal schooling; and as shown in compelling graphical depictions, it reinforces the centrality of beginning a trajectory of achievement during the early years of rapid growth. The paper suggests that a logarithmic characterisation of growth on a quantitative scale as a function of time in schooling unifies the understanding of the development in school mathematics and reading achievement. In particular, it solves the paradox that when comparisons are made in terms of grade equivalents, cohorts initially disadvantaged appear to become relatively even more disadvantaged over time, even though when comparisons are made in terms of rate of growth on a quantitative scale, the disadvantaged groups may have an equal or higher rate of growth.

Implications of a law of a decelerating logarithmic growth in mathematics and reading achievement

Introduction

Young children have a remarkable facility for learning skills, illustrated with languages, where the younger one begins learning a skill the greater the potential to become very proficient at it (Bloom & Sosniak, 1985). Sylva (1997) summarises studies of critical periods of learning, including their effects on IQ which seemingly can be enhanced by a stimulating environment (Keats, 1980). Likewise, research by Heckman and Kautz (2012) and Gustafsson et al. (2010) has shown the significance of acquiring social competence in early childhood. In literate societies, mathematical and literacy skills are essential for social competence. Although it is generally understood that beginning a trajectory of growth in early childhood in reading is vital (Paris, 2005), research shows that beginning a trajectory of growth in mathematics in early childhood is equally vital. Moreover, for both mathematics and reading proficiencies, there is no elegant characterisation of growth which reflects, confirms, and quantifies the importance of an early growth trajectory.

Instead, the modelling of growth is carried out typically using the methods of the social sciences, imported from statistics, in which competing linear models are tested against each other and where different rates of growth are seen to characterise different periods of growth. This approach is illustrated in the study of growth in mathematics in Lee (2012) and in reading in Francis et al. (1996). One exception is the modelling of growth in mathematics and reading proficiency using the non-linear Gompertz function used to describe growth in populations.

The Gompertz function involves three parameters in general has a sigmoid shape, and because it involves the exponential of an exponential, is relatively complex. This paper shows that it is possible to characterise the growth in reading and mathematics achievement from a number of studies in terms of a decelerating, near-perfect logarithmic function of the time spent in schooling. Such characterisation of a relationship, which involves only two parameters, is more akin to that in the natural than social sciences.

The decelerating growth is expected from both biological and intellectual variables (Bock, 1983). In addition, to the degree that such a growth function is confirmed, to that degree it that the chosen origin is compatible with the data, that the unit across the range of the scale is relatively constant, and that the variable assessed is quantitatively relatively unidimensional.

This unidimensionality does not preclude qualitatively distinct features in proficiency which are mapped onto the same dimension (Styles, 1999). In addition, although the unit might be the same at different parts of the continuum of the dimension, it is to be understood that the same difference across the dimension is constrained in meaning only quantitatively. Magnusson (1966) illustrates the point by contrasting the contextual implications of a 0.25 metres difference in competitive broad jumps between 4.25 and 4.50 metres and between 8.00 and 8.25 metres.

The decelerating growth function on a quantitative scale summarises efficiently and powerfully studies concerned with the importance of beginning a growth trajectory of mathematics and reading achievement in early childhood. In addition, it suggests that there may be a critical period for learning the elementary skills of mathematics and reading in early childhood. Finally, the logarithmic function provides constraints on the relative magnitude of growth that can be expected on a quantitative scale across different periods of schooling.

The rest of the paper is structured as follows. First, using two examples of published studies in each of mathematics and reading, growth is characterised by the logarithmic function. It is acknowledged, and taken for granted, that the construction of the data in terms of measurement properties on the same scale is sound and that the previous analyses of these studies were thorough. Second, a set of Australian data in both mathematics and reading, in which the growth trajectories of academically advantaged and disadvantaged groups are compared, are shown and interpreted.

A final preliminary comment is made regarding trajectories of cohorts and individuals. Growth at the group level smooths out individual differences which may include growth spurts (Tanner, 1962; Andrich & Styles, 1994). For the functions to be identical at the group and individual level, the principle of dynamic consistency would have to hold (Keats, 1980). However, the concern here is with macro-level group measurements, the kind used in monitoring mathematics and reading at national and jurisdictional levels. Although not as smooth as group means, individual growth must be consistent with the growth of group means in that it is also decelerating.

Mathematics examples

The mathematics examples summarised are from Lee (2012) and from Cameron et al. (2015), with both data sets from the USA. The former involves Grades 1 to 12 from an integration of three national, longitudinal data sets including the Early Childhood Longitudinal Study - Birth

Cohort (ECLS-B). Although there are data for prekindergarten and kindergarten, here we report means from Grade 1. The latter data, involving Grades 1 to 9, are from the National Longitudinal Survey of Youth—Children and Young Adults (NLSY) and the Early Childhood Longitudinal Study—Kindergarten Cohort. Comprehensive information about these respective datasets is provided in Lee (2012) and Cameron et al. (2015). In the above papers, the former data were analysed using linear models with a substantive concern for readiness for college, while the latter were analysed using a polynomial model, a logistic function and the Gompertz model.

Figure 1 shows the logarithm modelling of the Lee (ECLS-B) and Cameron et al. (NLSY) studies as a function of grade in school beginning with Grade 1 where the logarithmic function takes the form

$$y_g = a + b\ln(x_g) \qquad (1)$$

where y_g is the quantitative mean achievement of a cohort, in an arbitrary, constant unit with an arbitrary origin. at time x_g. Inthis paper is the x_g is the number of the grade in schooling beginning with Grade 1, and because $\ln(1) = 0$, a is the modelled achievement in Grade 1.

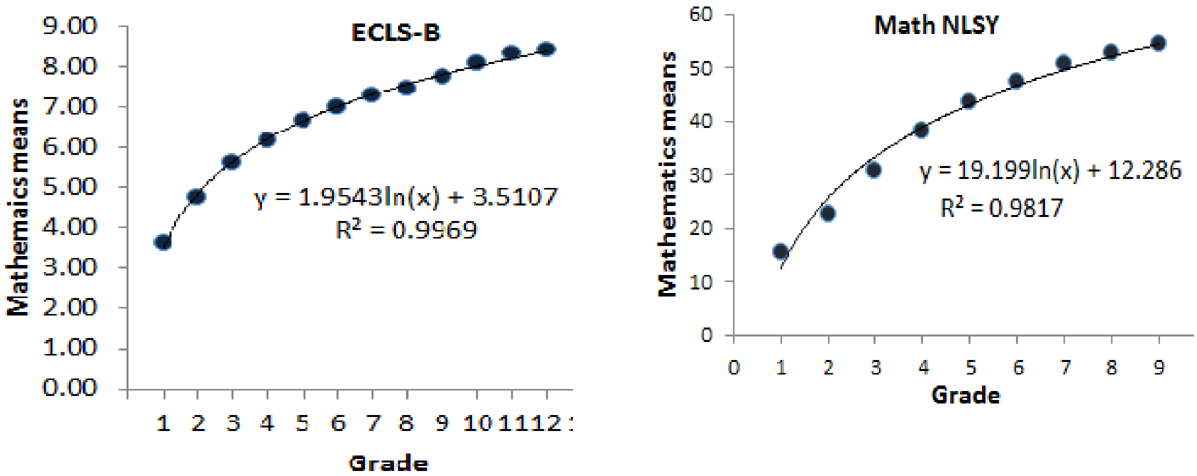

Figure 1. Logarithmically decelerating growth in mathematics as a function of grade.

Where there are data from kindergarten, it is considered that they may be less stable than those from the beginning of formal schooling. Where age information is provided, it is considered that age 6 is equivalent to Grade 1. Grades rather than age are chosen in part because growth is then interpreted as a function of time in formal schooling and because the fit is slightly better, suggesting it is a more consistent choice of origin. This choice is reconsidered later in the paper. The logarithmic function fits remarkably well, with $R^2 \geq 0.98$

in both examples. Although not presented below, other examples with similar high correlations for logarithmic growth in mathematics as a function of time in schooling are found in Mok et al. (2015) and Shanley (2016). In the NLSY example, there is a hint of an S shaped curve in the early grades consistent with the Gompertz model but it seems the complication relative to the interpretive gain is minimal. It may also be considered that the mean in Grade 1 is somewhat aberrant because if it is eliminated, the logarithmic function for the remaining means has $R^2 = 0.996$!

Reading examples

Examples of growth in reading, summarised in Figure 2, are from Francis et al. (1996) and from Cameron et al. (2015), which provide comprehensive information regarding the data. Both data sets are from the USA. The former involves data from non-reading impaired students in Grades 1 to 9 in the Connecticut Longitudinal Study (CLS) and the latter from the NLSY, part of the same study as the mathematics example. In the above papers, the former data were analysed using a model involving two equations that constrained the trend to approaching an asymptote, while the latter were analysed using a polynomial model, a logistic function and the Gompertz model. Again, the growth is presented as a function of grade.

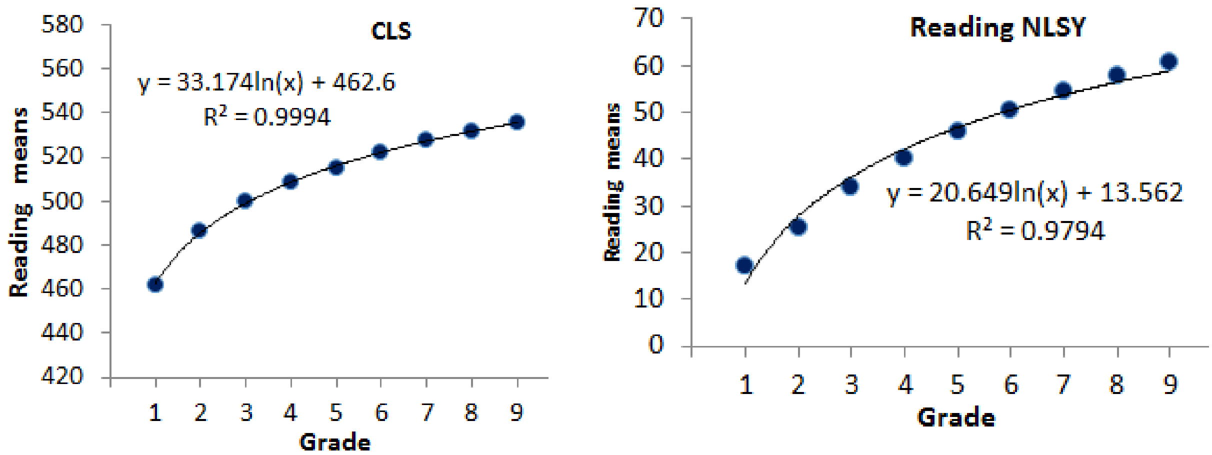

Figure 2. Logarithmically decelerating growth in reading as a function of grade.

It is evident that the logarithmic function again characterises growth remarkably well, with $R^2 \geq 0.98$ in both examples. Although not presented below, other examples with similar high correlations for logarithmic growth in reading as a function of time in schooling are found in Keiffer (2012) and Roberts et al. (2010). Again, in the NLSY example, there is a hint of an S shaped curve but the complication of the Gompertz model relative to the interpretive gain is again minimal. In addition, if Grade 1 is eliminated, the logarithmic function for the remaining means has $R^2 = 0.998$, suggesting that the mean of Grade 1 is aberrant.

Comparisons among educationally advantaged and disadvantaged groups

Figure 3 shows examples of growth from an educationally disadvantaged and advantaged group. Such examples show further power in the interpretation of logarithmic growth relative to those from other models. The data are measurements from mathematics assessments in the school Years 3, 5, 7 and 9 of the Australian National Assessment Program – Literacy and Numeracy (NAPLAN) where every student is assessed in Years (Grades) 3, 5, 7 and 9. The data shown are from the group that was assessed first in 2010 which may be found on the Australian Curriculum, Assessment and Reporting Authority (2017) website http://www.nap.edu.au/. The assessments across the grades and across years are linked to a common scale using Rasch measurement models (Rasch, 1960).

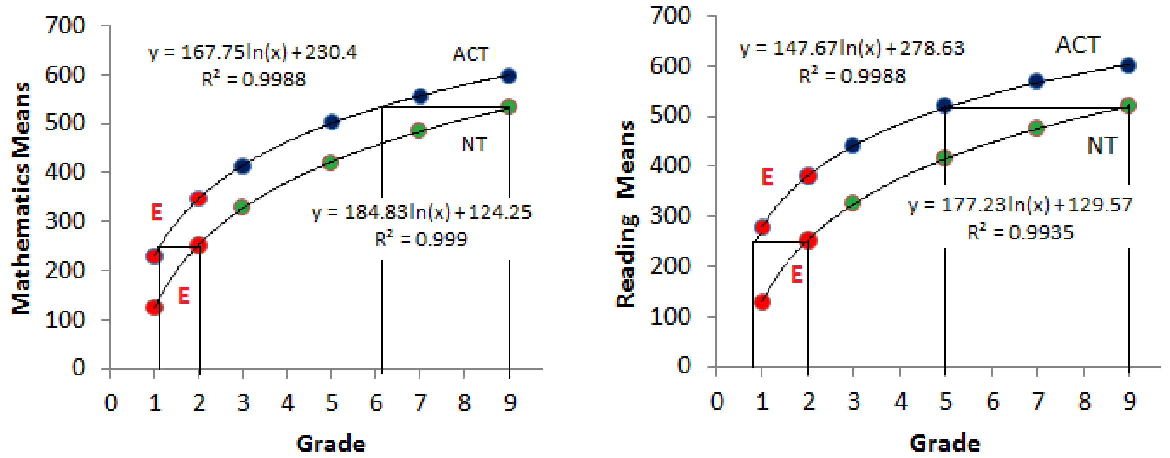

Figure 3. Logarithmically decelerating growth as a function of grade in two jurisdictions with extrapolated values (**E**) shown with dots (●) for Grades 1 and 2.

Figure 3 shows growth in Mathematics and Reading from two jurisdictions, the Northern Territory (NT) and the Australian Capital Territory (ACT). The former, comprised of a small population in many remote communities, shows consistent average educational disadvantage while the latter, comprised of a small population in the Australian capital, Canberra, shows consistent advantage. It is recognised that there are only four points of interpolation, giving only two degrees of freedom in the logarithmic model. However, the fit is near perfect in both cases with the growth clearly non-linear and decelerating. Although not shown here, the growth is also near perfect in both mathematics $(R^2 = 0.9996)$ and reading $(R^2 = 9995)$ for the means of students from all Australian jurisdictions. The graphs also show two points of extrapolation of achievement for Grades 1 and 2 shown in red (●). The extrapolation depends on the accuracy of the logarithmic function, but given evidence from earlier examples of the success of the logarithmic function to characterise growth from Grade 1 through to Grade 9 and beyond, the extrapolation seems justified.

Logarithmic function and decelerating growth

There are a number of advantages in using the logarithmic function, when it applies as well as it does in the above examples, in interpreting the nature of growth relative to other interpretations. To highlight these advantages, we briefly summarise two other interpretations.

Some interpretations of decelerating growth

First, the interpretation for research carried out by Lee is that *The emerging trends imply a tripartite pattern where American students are gaining ground at the pre/early primary school level, holding ground at the middle school level, and losing ground at the high school level*. (Lee, 2010, p.800).

Second, researchers in Australia have observed, but not explained, the relative growth in proficiency of advantaged and disadvantaged groups in Australia which are usually reported simply on a linear scale. *Converting NAPLAN data into years of progress provides striking insight into relative learning progress between Year 3 and Year 9. Low achieving students fall even further back. Low achievers in Year 3 are an extra year behind high achievers by Year 9. They are two years eight months behind in Year 3, and three years eight months behind by Year 9.* (Goss et al., 2016, p.2).

Third, in a footnote, Cameron et al. consider it *Worth noting … that deceleration, across various age spans, is one theme in the many inconsistencies across studies of longitudinal achievement*. (Cameron et al., 2015, p.791).

Each of the above commentaries implies some kind of complication in interpreting growth where it seems that by implication, linear growth is interpreted in terms of segments of time. We contrast this with the interpretation with that from logarithmic growth below.

There are two classes of advantages in characterising growth logarithmically - one quantitative, the other interpretive.

Quantitative advantages of the logarithmic growth

In terms of quantitative advantages, first, the logarithmic function characterises the growth across the years of schooling as only one function with just two parameters to be estimated, the implied intercept on the achievement scale and the rate of growth given by the coefficient of $\ln(grade)$. This has an elegance, simplicity, and parsimony, found and expected in the natural sciences, but rarely observed or expected in the social sciences.

Second, the simple function in which the $(R^2 > 0.98)$ suggests strongly that the quality of assessments and their transformation into measurements onto a common scale, that is, the same unit and origin, has itself been successful in the range of studies reported. It would be most unlikely that this simple, logarithmic function would be so accurate in accounting for growth without the unit being constant across the range of measurements. This is perhaps not surprising given the attention that this step has been given in each reported study. Moreover, this conclusion is confirmed, not only because the function is simple, but because the function shows an expected decelerating growth (Bock, 1983).

Third, the function is invariant with respect to the unit of the independent variable, in this case time in schooling. Thus if the grade were to be converted to months in schooling, then the rate of growth R^2 would remain identical. For example, if the grades in the above examples were divided by 2 to characterise semesters, the regression slope would remain constant.

However, if a constant were added, for example the Grades were replaced by age and the 1 (for Grade 1) were was replaced by 6 (for age 6), then successive ages with 7, 8, 9 and so on, the fit is likely to change. For example, in the Maths NLSY study, shown in the right graph of Figure 1, if this substitution is made then the R^2 drops from 0.9817 to 0.9738, but more importantly there is a slight, though systematic, pattern of deviation from the logarithmic curve. That beginning with Grade 1 shows excellent fit suggests that the choice of grade gives an accurate origin.

Fourth, differentiating the logarithmic function gives the rate of growth as

$$\text{Rate of growth}: b/x. \qquad (2)$$

It is clear that the rate of growth is inversely related to x, in this paper the number of years of formal schooling characterised by the grade. Thus although the growth rate in Grade 1 in Mathematics, in the unit of the scale, in the ECLS-B data (left graph of Figure 1) is 1.9543, it is only 0.19543 in Grade 10 in the same unit.

Interpretative advantages of logarithmic growth

In terms of interpretive advantages, first, the simple function of decelerating growth unifies a number of different interpretations of growth across the continuum. For example, regarding Lee's interpretation quoted above, the interpretation is not that the students are "losing ground" at the high school level, but that the inevitable deceleration on a quantitative scale slows the growth.

Second, the rapid early growth, which decelerates on a quantitative scale, shows quantitatively the magnitude of the early growth and implies the importance of this period.

Third, by comparing the trajectories of growth of advantaged and disadvantaged groups, it provides an explanation as to why the *low achieving students fall even further back* as interpreted by Goss et al. above. Thus from growth trajectories of the ACT (advantaged) and NT (disadvantaged) in mathematics in Figure 3, it is evident that in terms of grade equivalents, the mean of NT students in Grade 2 is equivalent to that of the ACT students in Grade 1, a lag of 1 grade; however the mean of NT students in Grade 9 is equivalent to that of the ACT students in Grade 6, a lag of 3 grades. However, the quantitative rate of growth of the NT students (184.83) is greater than that for the ACT students (167.75). The apparent increasing difference is a direct result of the decelerating growth. For the growth in reading, the difference is less than one grade in Grade 1, to being four grades by Grade 9. Comparisons provided by Francis et al. (1996) between low achieving students and those not reading impaired shows the same features as the Australian data. This explanation of the apparent increasing difference on the grade scale does not minimise its substantive importance. Indeed it leads to the fourth advantage of the logarithmic growth curve.

Fourth, the growth curves in Figure 3 depicted graphically and compellingly that it is important to begin a growth trajectory as early as possible. It is evident from the extrapolated values to Grade 1 in both mathematics and reading that already in Grade 1 the NT students are disadvantaged compared to the ACT students. This is primarily the effect of home background compounded by opportunities to attend kindergarten.

Summary and conclusion

This paper demonstrates that growth in means of cohorts in mathematical and reading proficiency on a measurement scale is almost perfectly logarithmic with a very fast growth in the early years, and a decelerating growth over time. The rate of growth is inversely proportional to the time spent in schooling. It is surprising that this function has not been noticed before and that instead many models are used to describe the details of the growth. These complications in interpretation have the potential to lose sight of the point of the rapid early but decelerating growth. The need for early intervention to begin a trajectory of growth in mathematics and reading as early as possible in childhood is depicted graphically and vividly by the logarithmic growth function.

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